J. Fluid Mech. (1991), *vol.* 229, *pp.* 417–426
Printed in Great Britain

Barotropic instability of the Bickley jet

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(Received 17 September 1990 and in revised form 23 January 1991)

The linear stability of the zonal shear flow $\bar{u} = -\operatorname{sech}^2 y$ is investigated in the framework of the beta-plane approximation. This retrograde jet is known to be more unstable than its eastward-propagating counterpart and has some surprising characteristics. First, this is a rare example of a flow in which barotropically unstable modes occur that do not have a critical point. Secondly, singular neutral modes exist in which the critical point occurs at the centre of the jet, where $\bar{u}'_c = 0$. It is shown in this paper that such singular modes form part of the stability boundary both for the varicose mode and also for the radiating sinuous mode.

1. Introduction

The observed instability of zonal currents in planetary atmospheres has motivated numerous studies of the barotropic stability of such flows. Most of this work has been based on the linear theory and its relevance to the Earth's atmosphere at tropical latitudes is discussed, for example, in the survey article by Kuo (1973). The theory is relevant to the ocean, as well, and Philander (1976) attributes turbulence in the equatorial surface currents of the Atlantic and Pacific oceans to barotropic instability. More recently, Hurlburt & Thompson (1980) have concluded on the basis of numerical simulations that certain eddy shedding phenomena in the Gulf of Mexico are due to barotropic instability of the jet-like Loop Current.

The basic equation describing the evolution of the flow is the vorticity equation, which can be written

$$\omega_t + \psi_y \omega_x - \psi_x \omega_y - \beta\psi_x = Re^{-1}\nabla^2\omega, \tag{1.1}$$

where the vorticity $\omega = -\nabla^2\psi$, Re is the Reynolds number and the velocity components are related to the stream function by $(u, v) = (\psi_y, -\psi_x)$. In the beta-plane approximation, the Coriolis effect is modelled by a linearization about some mean latitude and β is the derivative of the Coriolis parameter (assumed constant).

When $\beta = 0$, the unstable and neutral modes in the linear stability problem associated with (1.1) decay exponentially as $|y| \to \infty$. However, with $\beta > 0$, even in the absence of a mean flow, westward-propagating Rossby waves exist. If a mean flow is present whose velocity overlaps the phase speed of Rossby waves at a value of β where linear instability is possible, then the eigenfunctions may be oscillatory instead of exponentially decaying. These are the so-called radiating modes which must satisfy the condition of outward energy propagation, instead of exponential decay, at infinity. McIntyre & Weissman (1978) point out the significance of the phase speed condition and discuss other pertinent aspects of radiating instabilities.

Talley (1983) has argued that such radiating modes are needed to explain the large-scale eddy energy distribution outside zonal currents in mid-latitude oceans. Although the amplification rates are smaller for these modes, they have large meridional decay scales in agreement with observation. She has applied the linear

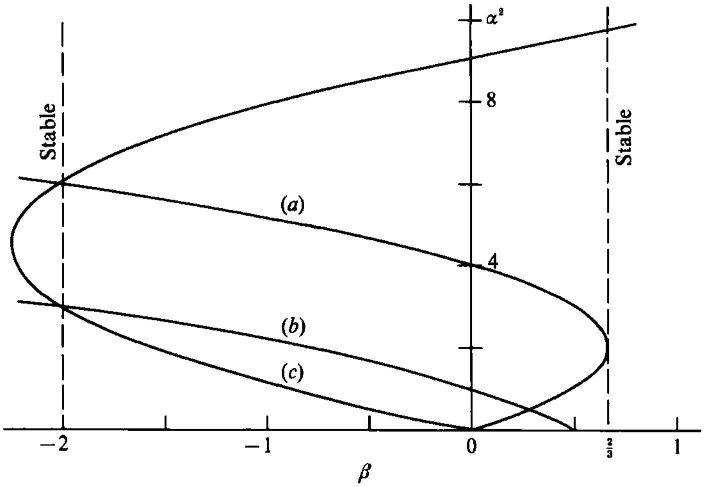

FIGURE 1. Neutral curves and stable regions for the jet $\bar{u} = \operatorname{sech}^2 y$. (*a*) Neutral solution for the sinuous mode. (*b*) Neutral solution for the varicose mode. (*c*) The singular neutral mode with $c = 1$.

theory to broken-line profiles modelling jets and mixing layers and finds the results to be qualitatively consistent with data for the western North Atlantic. Similar considerations have motivated a number of meteorological studies, such as the article by Dickinson & Clare (1973), who investigated both exponentially decaying and radiating unstable perturbations to a $\tanh y$ shear layer.

We consider small-amplitude disturbances to a mean parallel shear flow $\bar{u}(y)$ and employ (1.1) as the starting point. For perturbations whose stream function is of the form $\hat{\psi} = \phi(y) \exp\{i\alpha(x-ct)\}$, the linear, inviscid theory revolves around the Rayleigh–Kuo equation

$$(\bar{u}-c)(\phi''-\alpha^2\phi)+(\beta-\bar{u}'')\phi = 0, \tag{1.2}$$

where α is the wavenumber and c is a complex constant whose real part is the phase speed. As noted by Howard & Drazin (1964), although β is always positive, changing its sign is mathematically equivalent to reversing the flow direction; hence, it is to be understood that the results presented below for $\beta < 0$ and $\bar{u} = \operatorname{sech}^2 y$ correspond, in reality, to the retrograde jet $\bar{u} = -\operatorname{sech}^2 y$ with $\beta > 0$.

Two neutral modes for the Bickley jet were found by Lipps (1962), namely the sinuous mode

$$c = \tfrac{1}{6}\alpha^2, \quad \beta = \tfrac{1}{6}\alpha^2(4-\alpha^2), \quad \phi = \operatorname{sech}^2 y \tag{1.3}$$

and the varicose mode

$$c = \tfrac{1}{6}(3+\alpha^2), \quad \beta = \tfrac{1}{6}(1-\alpha^2)(3+\alpha^2), \quad \phi = \operatorname{sech} y \tanh y. \tag{1.4}$$

Howard & Drazin found, in addition, the singular neutral mode

$$c = 1, \quad \beta = -\tfrac{1}{9}\alpha^2(9-\alpha^2), \quad \phi = (\operatorname{sech} y)^{\alpha^2/3}(\tanh y)^{2-\alpha^2/3}. \tag{1.5}$$

The interpretation of this mode is not clear owing to the branch point at $y = 0$ and it will be discussed further in §3. In figure 1, the three neutral curves (1.3)–(1.5) are displayed.

From a generalization of Rayleigh's inflexion point theorem, it follows that the quantity $(\beta-\bar{u}'')$ must change sign somewhere for instability to occur; stability is therefore guaranteed for $\beta < -2$. However, because $\beta > -2$ is not a sufficient condition for instability, it cannot be concluded that $\beta = -2$ forms part of the stability boundary. Lipps, on the other hand, used Lin's perturbation formula to show that for $\beta > -2$ the neutral modes (1.3) and (1.4) do constitute stability boundaries. The missing portion from $\beta = -2$, $\alpha^2 = 6$ (or $\alpha^2 = 3$ in the case of the varicose mode) must be computed numerically. In much of this paper, the term 'neutral curve' refers to this part and it will be shown in §2 that the presence of radiating modes results in the stability boundary for the sinuous mode being quite complicated. It consists of three distinct portions and contains a cusp where the neutral curves for the radiating and exponentially decaying modes intersect.

For a given positive value of β, the maximum amplification factor αc_i is obtained by considering the sinuous mode and therefore most numerical computations have been done for that mode. However, there are two reasons why the varicose mode would appear to merit further attention. First, it could play a significant role in the nonlinear problem, possibly in the context of modal interactions. Leib & Goldstein (1989), for example, have studied the resonant interaction between sinuous and varicose modes in the $\beta = 0$ case. Secondly, it was found by Howard & Drazin that for long waves the varicose mode had larger growth rates. Although this is clearly the case for $\beta > 0$, our results presented in §3 do not support their conclusion for negative values of β. Interestingly, though, it is found that the singular solution (1.5) comprises the lower stability boundary for the varicose mode. We return to this topic in §3, but will first elucidate the stability properties of the sinuous mode in the following section as it appears to be the most important in practice.

2. Stability characteristics of the sinuous mode

In his survey article treating the general topic of barotropic instability, Kuo (1973) has presented numerical calculations for the unstable sinuous mode. These calculations are accurate for moderately unstable waves, but are in error for weakly amplified waves when β is negative. Kuo also concluded that the singular mode (1.5) forms part of the stability boundary for $\beta < -1$, but it will be seen below that such is not the case.

Subsequently, in a more thorough numerical investigation, Deblonde (1981) computed a portion of the missing stability boundary, namely, that part which begins at $(\beta, \alpha^2) = (-2, 6)$. This computation is made difficult by the fact that c is not far from unity along this curve so it was not clear *a priori* if the neutral modes were singular. Moreover, the presence of a continuous spectrum of modified Rossby waves to the left of the stability boundary makes it necessary to identify some property that distinguishes those neutral modes comprising the stability boundary from the ordinary modified Rossby waves.

It is, in fact, the non-analytic behaviour of the dispersion relation along the neutral curve that can be used to locate it precisely. Essentially, two modified Rossby waves coalesce, as β is increased, to become a single unstable mode once the stability boundary is crossed. A procedure exploiting this observation is to compute c as a function of β at constant α (or as a function of α with β constant) for the lowest Rossby mode and, if this curve has a minimum, it corresponds to a point on the stability boundary. The rationale behind this procedure is outlined in Drazin, Beaumont & Coaker (1982) and is essentially that one is looking for a point on the neutral curve c_0, say, where the coefficient of $(c-c_0)$ vanishes in the Taylor series expansion of the dispersion relation. At such a point, the dispersion relation will yield

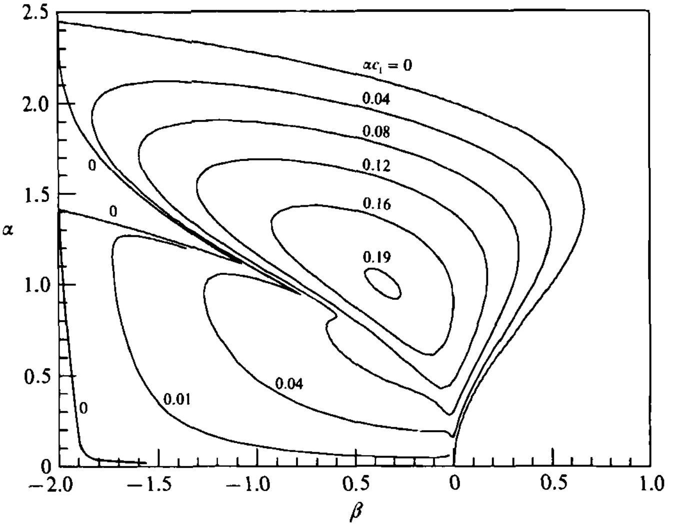

FIGURE 2. Curves of constant amplification rate for the sinuous instability mode.

an equation for $(c-c_0)^2$ and instability results if $(c-c_0)^2$ is equal to a negative quantity. The foregoing procedure was used to compute the portion of the neutral curve in figure 2 which begins at $(\beta, \alpha) = (-2, \sqrt{6})$ and ends at about $(\beta, \alpha) = (-1.06, 1.1)$.

A cusp is formed where this curve intersects a second neutral curve, found by the author, which is given analytically by

$$c = \tfrac{1}{6}(\alpha^2+4), \quad \beta = -\tfrac{1}{6}\alpha^2(\alpha^2+4), \quad \phi = 3\tanh^2 y - 1. \tag{2.1}$$

The method used to find (2.1), which also yields (1.3) and (1.4), was to transform (1.2) to the associated Legendre equation and then follow the procedure described in Mathews & Walker (1970, pp. 21–22).

The solution (2.1) for $-2 \leqslant \beta \leqslant 1.06$ comprises the upper stability boundary in figure 2 for the unstable radiating modes as the vertical wavenumber vanishes (note that ϕ is neither oscillatory nor exponentially decaying as $|y| \to \infty$, but that ϕ' vanishes). Interestingly, Talley (1983) also found neutral curves with a cusp and she speculated that this was due to the discontinuities in the velocity profiles she employed. Now, it is clear that it is rather the presence of both radiating and exponentially decaying modes of instability that is responsible. A similar phenomenon occurs in the marginal stability curve for the compressible $\tanh y$ shear layer investigated by Blumen, Drazin & Billings (1975) where, again, more than one mode of instability is present.

Turning now to the numerical results for $c_i > 0$, a rather surprising result found by Deblonde was that weakly amplified modes that do not have critical layers as $c_i \downarrow 0$ occur in the region toward the upper left of the stability diagram in figure 2. (See, also, the modified Rossby wave results in figure 3 of Drazin *et al.* which are completely consistent with Deblonde's computations.) Although the existence of such modes is permitted by a modified version of the semicircle theorem due to Pedlosky (1964), no other examples where this occurs are known to the author.† In fact, Tung (1981) has proved a theorem which would prohibit these modes because it states that neutral modes adjacent to unstable waves without critical points must propagate with a

† A second example I have just come across is the sinusoidal jet studied by Yamasaki & Wada (1972).

speed c lying within the range of $\bar{u}(y)$. However, Tung's proof assumes that the dispersion relation is analytic on the stability boundary, but that is not the case on this portion of the neutral curve owing to the coalescence of modified Rossby waves. Collings & Grimshaw (1984) discuss this phenomenon in the context of barotropic shelf waves, where the governing equation is a generalization of (1.2).

The curves of constant αc_i were found by both Kuo and Deblonde to be smooth for $\alpha c_i \geqslant 0.10$. However, Deblonde found that a sort of kink reversal begins to form in the curve for $\alpha c_i = 0.08$. It was later realized that this was due to the overlap of the radiating and 'trapped' modes. For smaller values of αc_i the curves of constant αc_i characteristically reverse direction, as illustrated in figure 2 by the curve for $\alpha c_i = 0.04$.

Given that certain results obtained by Deblonde were unexpected and that there was some disagreement with those of Kuo, a few words about her numerical procedures (and those we have used to extend her results) are appropriate. Both finite-difference and initial-value (shooting) methods were employed, as well as coordinate transformations to make the integration domain finite. When c_i was small, the contour of integration was deformed into the complex plane in accordance with the analysis of Foote & Lin (1950) so that it was usually not necessary to pass near a singularity. (However, there are regions in parameter space where the two critical points approach each other and a very small step size must then be utilized.) It was verified that programs employing different methods gave the same results and agreement with the long-wave, small-β expansions of Howard & Drazin was very good in the domain where the latter are valid.

2.1. *Singular radiating modes*

It remains to discuss the portion of the stability boundary that descends from the point $(\beta, \alpha) = (-2, \sqrt{2})$ which consists of singular radiating modes with $c = 1$. First, we observe that Reynolds stress considerations require that these modes be singular. The Reynolds stress τ for a neutral mode is constant in y and, if $\phi(y)$ is even and oscillatory as $|y| \rightarrow \infty$, then the value of τ will be a different non-zero constant above the jet than below. There seems no obvious reason to rule out the possibility of $c < 1$, in which case, there would be jumps in τ across two critical points with $\tau = 0$ between them. However, a careful numerical study of the unstable radiating modes as $c_i \downarrow 0$ left no doubt that $c = 1$ on the stability boundary. Hence, the Reynolds stress distribution is as illustrated in figure 3.

When $c = 1$, $\bar{u}'_c = 0$ and the Frobenius solutions valid for small y have the same form as in a stratified shear flow with Richardson number less than $\frac{1}{4}$, namely

$$\phi(y) = A|y|^{\frac{1}{2}+\lambda}\left\{1+\frac{\alpha^2+\frac{2}{3}\beta-\frac{20}{3}}{4(1+\lambda)}y^2+\ldots\right\}+B|y|^{\frac{1}{2}-\lambda}\left\{1+\frac{\alpha^2+\frac{2}{3}\beta-\frac{20}{3}}{4(1-\lambda)}y^2+\ldots\right\}, \quad (2.2)$$

where $\lambda = (\beta+\frac{9}{4})^{\frac{1}{2}}$. The Reynolds stress is given by

$$\tau = \tfrac{1}{2}\alpha(\phi'\phi^*)_i, \quad (2.3)$$

where i denotes the imaginary part, and for $y > 0$ we obtain

$$\tau = \lambda\alpha(AB^*)_i. \quad (2.4)$$

This result is equivalent to equation (5.11) of Miles (1961) for the stratified case. However, the conclusion we will draw from (2.4) is the opposite. According to theorem VIII of Miles (1961), either A or B must be zero for a singular neutral mode

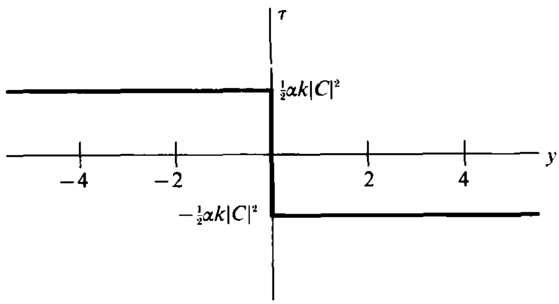

FIGURE 3. Variation of the Reynolds stress for a sinuous, radiating neutral mode.

because the Reynolds stress vanishes at the boundaries in the cases he considered. Here, by contrast, both A and B must be non-zero in order to permit the jump in τ across the critical point indicated in figure 3.

Far away from the jet, $\bar{u}''$ and $\bar{u}$ vanish, so that, as $y \rightarrow \pm\infty$,

$$\phi_{\pm} \sim C\mathrm{e}^{\mp \mathrm{i}ky}, \quad \text{where} \quad k = \left(-\frac{\beta}{c} - \alpha^2\right)^{\frac{1}{2}}. \tag{2.5}$$

The signs in the exponential have been determined by imposing the radiation condition that the group velocity $\partial\omega/\partial k$ be outward as $|y| \rightarrow \infty$, where $\omega = \alpha c$. In the case of the Bickley jet,

$$\frac{\partial \omega}{\partial k} = \frac{2\alpha\beta k}{(k^2+\alpha^2)^2}. \tag{2.6}$$

The corresponding analysis for radiating modes in a $\tanh y$ shear layer was formulated by Hickernell (1984). That problem differs from the present one in that the solutions are radiating on only one side of the shear layer. Consequently, a critical layer is required across which τ jumps from zero on the side where ϕ has exponential decay to a non-zero constant on the radiating side.

Returning now to the Bickley jet, we note that (2.3)–(2.5) can be used to obtain a relationship between the constants A, B and C (one of which is arbitrary). Specifically, owing to the symmetry of ϕ, it is true that on either side of the jet

$$\lambda\{AB^*\}_{\mathrm{i}} = \tfrac{1}{2}k|C|^2. \tag{2.7}$$

This relationship was employed as a check in our numerical procedure for computing eigenfunctions for neutral, radiating modes, which was the following.

The integration was initiated at $y = 0.05$ using the Frobenius expansion (2.2) to obtain initial values for ϕ and ϕ'. Then, (1.2) was integrated out to $y = 3$ using a fourth-order Runge–Kutta method. Two solutions denoted $\phi_A(A = 1,\ B = 0)$ and $\phi_B(A = 0,\ B = 1)$ were obtained in this way and then superimposed so as to satisfy the radiation condition at $y = 3$, namely

$$\phi' + \mathrm{i}k\phi = 0. \tag{2.8}$$

The arbitrary constant was chosen to be $A = 1$, B was calculated by imposing (2.8) and the value of C determined from the result was then substituted into (2.7) to verify the procedure. Neutral values of α and β were obtained by extrapolating results for unstable modes with very small growth rates (on the order of $\alpha c_{\mathrm{i}} = 10^{-3}$). A typical eigenfunction is illustrated in figure 4.

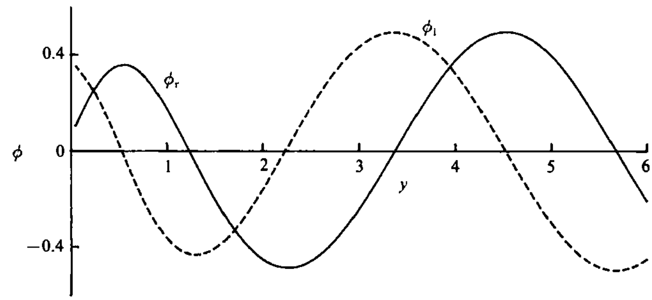

FIGURE 4. Eigenfunction for a neutral radiating mode; $c = 1$, $\alpha = 0.3$ and $\beta = -1.94$.

3. The varicose mode and the singular non-radiating mode

The varicose mode has received little attention in the literature because it has lower amplification rates when $\beta = 0$ and it is unstable for a smaller range of wavenumbers when $\beta \neq 0$. However, it is clear from figure 1 that for $\beta > 0$ the varicose mode is more unstable when α is small and according to the long-wave analysis of Howard & Drazin it has some unusual properties when β is negative. In particular, in the limit $\alpha \to 0$ with β/α^2 fixed, there is a neutral curve $\beta = -\alpha^2$. On either side of this neutral curve, instability is predicted and, for $-\beta > \alpha^2$, Howard & Drazin find

$$c \sim 1 + \mathrm{e}^{\pm \mathrm{i}\pi/3} \{\tfrac{1}{4}\pi\alpha(-1-\beta/\alpha^2)^{\frac{1}{2}}\}^{\frac{2}{3}} \tag{3.1}$$

so that there can be instability with $c_r > 1$.

The possibility of instability on either side of a neutral curve, which occurs in Charney's model of baroclinic instability, is also suggested by the equivalence pointed out by Lindzen, Rosenthal & Farrell (1983) between the barotropic instability of a point jet and the baroclinic problem posed by Charney. Our results, however, illustrated in figure 5, failed to yield the instabilities predicted by (3.1). Although modes with $c > 1$ were obtained in the region $-\beta > \alpha^2$, they had $c_i = 0$ and, hence, are modified Rossby waves rather than barotropic instabilities. The numerical computations of Kwon & Mak (1988) for a bounded Bickley jet also failed to produce instabilities in the region in question. They attributed this to crude numerical resolution, but our calculations with much greater resolution yield the same result. It must be concluded then that the + sign in (3.1) corresponds to taking an inadmissible root when inverting the long-wave expansion to solve for c. This result was disappointing because (3.1) seemed to offer an explanation for some recent experimental observations of Sommeria, Meyers & Swinney (1991) which indicated that for a retrograde jet the varicose mode was the most unstable.

These results illustrate the limitations of broken-line profiles because the top-hat jet is unstable for all negative values of β in the region $a \equiv \beta/\alpha^2 < -1$. This occurs because of the infinite vorticity at the velocity discontinuities, which cannot be counteracted by the stabilizing beta effect as it is in the case of the Bickley jet where the flow is stable for $\beta \leqslant -2$. For long waves, the dispersion relation of the top-hat jet yields the unstable solution

$$c \sim 1 + \alpha^{\frac{1}{2}} 2^{-\frac{1}{2}} (-1+\mathrm{i})(-1-a)^{\frac{1}{4}} + \dots \tag{3.2}$$

which agrees with (3.1) in predicting instability, but now with $c_r < 1$. (The corresponding small-α result in Howard & Drazin is in error, but their general dispersion relation (5.3) and expansions for $a > -1$ and $a = 1$ are correct.)

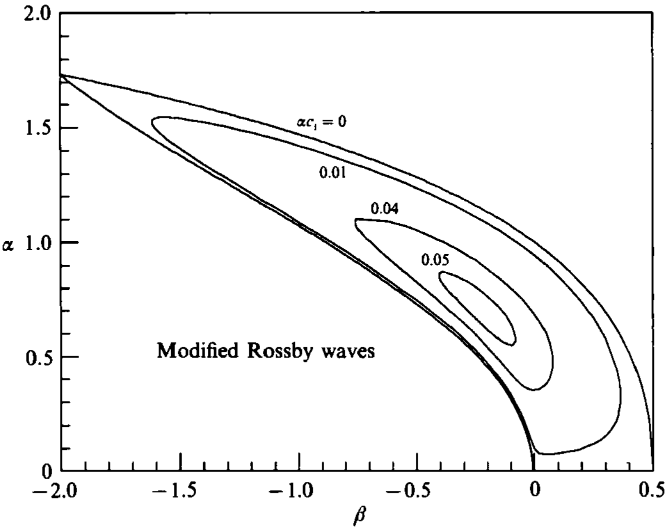

FIGURE 5. Curves of constant amplification rate for the varicose instability mode.

The numerical solutions in figure 5 did reveal an interesting and surprising result relative to the singular mode (1.5). It can be seen that this solution yields the lower stability boundary for the varicose mode, whereas Howard & Drazin (1964) concluded that (1.5) was a sinuous mode that did not form part of a stability boundary. Not only does the curve for $\alpha c_i = 0.01$ follow closely the neutral solution (1.5) for $\beta(\alpha)$, but c_r is very close to unity [e.g. at $(\beta, \alpha) = (-1.30, 1.275)$, $c_r = 0.987$], so there is no doubt that (1.5) is part of the stability boundary.

Because the singularity in (1.5) is of the same form as that encountered in stratified shear flows and $\phi \rightarrow 0$ as $|y| \rightarrow \infty$, the theorem due to Miles cited below (2.4) applies here, i.e. ϕ must be proportional to one or the other of the Frobenius solutions. For $\alpha^2 < \frac{9}{2}$ it is proportional to ϕ_A, whereas for $\alpha^2 > \frac{9}{2}$ it is proportional to ϕ_B.

The continuation of ϕ across the branch point at $y = 0$, however, cannot be deduced from previous studies of stratified shear flows because $\bar{u}_c' = 0$ in the present case. A critical-layer analysis of the Orr–Sommerfeld equation as $Re \rightarrow \infty$ is one possibility and this is being pursued by the author in collaboration with Professor S. N. Brown. Because $\bar{u}_c' = 0$ now, the critical-layer thickness is $(\alpha Re)^{-\frac{1}{4}}$ instead of the usual $(\alpha Re)^{-\frac{1}{3}}$. Introducing the inner variables

$$\eta = (\alpha Re)^{\frac{1}{4}} y \quad \text{and} \quad \chi(\eta) = \phi(y),$$

the governing equation in the critical layer is now found to be

$$\chi^{\mathrm{iv}} + \mathrm{i}\eta^2\chi'' - \mathrm{i}(\beta+2)\chi = 0. \tag{3.3}$$

Preliminary indications are that solutions of (3.3) are compatible only with an outer solution dominated by ϕ_A, in which case we must have $\alpha^2 \leqslant \frac{9}{2}$ (the regular solution at $(\beta, \alpha^2) = (-2, 6)$ is an exception). This conclusion seems to agree with the inviscid approach in §4 of Drazin *et al.* (1982) based on an analysis of modified Rossby wave solutions of (1.2) in the limit $c \downarrow 1$.

Some as yet unpublished neutral solutions of the Orr–Sommerfeld equation found numerically by Mr A. G. Burns indicate that as $Re \rightarrow \infty$, c is greater than unity. Hence, the limit $c_i \downarrow 0$ and $Re \rightarrow \infty$ of the Orr–Sommerfeld equation is non-uniform and the result for an inviscid singular neutral solution depends on the order in which the limits are taken.

4. Concluding remarks

The stability boundary for the sinuous mode of instability in the case of a retrograde jet was shown in §2 to consist of radiating modes for relatively long waves, i.e. $\alpha^2 \leqslant 2$, and modes that decay exponentially as $|y| \to \infty$ for larger wavenumbers. For a given value of β the radiating modes have smaller amplification rates but they are, nonetheless, believed to be significant in describing phenomena where the meridional scales are large. A third portion of the neutral curve comprised of modes that are bounded at infinity but whose eigenfunctions neither radiate nor decay exponentially was obtained in closed form (see (2.1)). The latter neutral curve forms a cusp where it intersects the stability boundary for the exponentially decaying modes. Such complex behaviour is likely to be typical of many shear flows when planetary rotation, density stratification or compressibility effects are taken into account.

Although the present investigation was based entirely on the inviscid theory, a number of questions arose suggesting that a study of the corresponding Orr–Sommerfeld problem would be worthwhile. For example, singular modes with a critical point at the jet maximum occur for both the radiating and bound states. In addition, inviscid neutral solutions found previously by Lipps (1962) and Howard & Drazin (1964) exist for $\beta < -2$, where the flow is stable, and one wonders what the effect of viscosity would be on these modes as well as the various modified Rossby waves computed here and by Drazin *et al.* (1982).

The radiating sinuous modes in figure 2 also are likely to be affected significantly by viscosity. For example, our numerical solutions of the Orr–Sommerfeld equation indicate the presence of a stability boundary at small values of α which is absent in the inviscid problem (where c_i is greater than zero when $\alpha = 0$). This neutral curve corresponds to a viscous mode, i.e. as $Re \to \infty$, $\alpha \to 0$ such that αRe is constant and the viscous terms in the Orr–Sommerfeld equation do not vanish. There is also some evidence of a neutral curve in the sinuous case related to the singular mode (1.5); clearly, the modal structure of the viscous problem is quite complex.

A recent study of the inviscid initial-value problem for the parabolic jet $\bar{u} = \frac{1}{2}\beta y^2$ by Brunet & Warn (1990) again suggests that solutions with critical layers at the jet maximum are significant. Their asymptotic solution as $t \to \infty$ indicates the formation of a nonlinear critical layer of thickness $O(\epsilon^{\frac{1}{2}})$ when $t \sim O(\epsilon^{-1})$ or longer, where ϵ is an amplitude parameter. This singular behaviour is not found when the velocity profile is monotone so it is interesting and surprising to obtain such a result for a jet profile that without external forcing does not even admit modal solutions.

Finally, we remark that a natural extension of the present analysis would be to treat weakly nonlinear disturbances now that the details of the stability boundary are known for the linear problem. A number of such studies have been carried out for the $\tanh y$ shear layer (see e.g. Churilov & Shukhman 1987) but none have been reported for the Bickley jet. Owing to the non-analytic behaviour along portions of the stability boundary, a variety of amplitude evolution equations would be obtained, so such a study could be quite interesting.

The author is indebted to Professor Tom Warn for illuminating discussions concerning radiating modes and other aspects of this work. He is also grateful to Mr Claude Beauchamp for his competent assistance with the computational work reported herein. Finally, the research support through grants INF-7939 and OGP-8764 of the Natural Sciences and Engineering Research Council of Canada is acknowledged, as well as the support of an FCAR grant from the Ministère de l'Education du Québec.

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